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HYDRODYNAMIC RELATIONS IN SUPERCONDUCTIVITY

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We explore the consequences for macroscopic effective Lagrangians of assuming that the momentum density is proportional to the flow of a conserved current. Such a relation holds in models that microscopically contain only one type of charge carrier. We show that such a relation leads to certain universal corrections to the usual effective Lagrangians considered for the description of superfluids. A fully consistent derivation of the superconducting Hall effect predicted by London is given. Some related issues that arise in anyon superconductivity are discussed: the linear Hall effect is derived in a transparent way, and the vanishing of the b term (under appropriate assumptions) is demonstrated formally. Some other possible applications are sketched.

1. Introduction

It is a common procedure in physics to form effective Lagrangians to describe the low-energy dynamics of interacting systems. This is useful when at low energies there are rather few degrees of freedom capable of being excited. These degrees of freedom are then described by quasiparticle fields, and their interactions are described by an effective Lagrangian written in terms of these quasiparticle fields. Typically the effective Lagrangian is truncated by retaining only a few terms of low order in temporal and spatial gradients, as is appropriate for an approximate description at small energies and momenta. The terms retained in the effective theory must exhibit the symmetries of the underlying microscopic theory either explicitly, or, in the case of spontaneous symmetry breaking, in the Nambu-Goldstone or Higgs modes.

Some symmetries of the microscopic theory may be realized trivially, with all the quasiparticle fields transforming as singlets — this is a form of confinement. Another class of constraints on the effective theory comes from the requirement that they match anomalous Ward identities of the microscopic theory.[1]

In this paper we shall exemplify yet another type of constraint that may arise on effective Lagrangians. It is the requirement that *algebraic identities* among the operators implementing symmetries that hold in the microscopic theory must be maintained in the effective theory.

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The specific identity we shall consider here is

$$J_i = \frac{e}{m} T_{0i} \tag{1.1}$$

where T_{0i} is the momentum density and J_i is the density of a conserved current. This identity has a simple interpretation. It states, for a Galilean invariant system, that all the momentum, and all the current, is carried by a single species of particle with charge-to-mass ratio e/m.

Since hydrodynamics embodies the laws of momentum flow, the consequences of (1.1) can be considered as consequences of hydrodynamics for *current* flow. In the context of superfluidity and superconductivity, this equation may be considered as the formal expression of the hydrodynamic analogies sometimes employed in these subjects.

We will now describe two typical examples of systems obeying (1.1). The first is a simple model of a BCS-type superconductor, and the second is an exotic "anyon" superconductor.[2–7]

For a simple model of a BCS superconductor, consider a spin one-half Fermi field ψ_α ($\alpha = 1,2$ labels the spin degrees of freedom). Consider the Lagrangian

$$\mathscr{L} = \int dt \int d^n x \left(\psi_\alpha{}^* i\partial_0 \psi_\alpha - \frac{1}{2m} \partial_i \psi^*{}_\alpha \partial_i \psi_\alpha + \lambda (\psi_\alpha{}^* \psi_\alpha)^2 \right). \tag{1.2}$$

This system is not exactly soluble. However, for $\lambda > 0$, the interaction term is attractive, and familiar arguments indicate that in the presence of a non-zero chemical potential, the many-body system with Lagrangian (1.2) is a BCS-type superconductor. In fact, (1.2) might well be regarded as a minimal model of BCS superconductivity. For our purposes, the interest of (1.2) focuses on the fact that — as one may easily verify — the relation (1.1) is obeyed by this system, if J_i is the conventional conserved current associated with the conserved charge $Q = e \int d^n x\, (\psi_\alpha{}^* \psi_\alpha)$. The physical reason for this is that all fields entering (1.2) have the same charge to mass ratio.

Our second example of a system obeying (1.1) is a two-dimensional anyon gas. One considers in two spatial dimension a system of identical particles (conserved in number and with no internal quantum numbers) of fractional statistics, with no interaction except that which is implicit in the statistics. As described at length in Ref. 3, such a system obeys (1.1), essentially because of the common charge to mass ratio of the anyons.

For values of the parameters for which these systems are superconductors (for positive λ, in the first case, and for certain values of the statistics, in the second case), they can be described at long wavelengths in terms of a single massless scalar field ϕ, which is a conventional Goldstone boson in the first case and plays a somewhat similar role in the second case. Our challenge in this paper is to understand how, in terms of such a massless scalar field, one can implement both an underlying symmetry and the relation (1.1). It is well known that the former can be implemented by requiring the phenomenological Lagrangian to be invariant under $\phi \rightarrow \phi + \text{constant}$, but the implementation of the additional relation (1.1) by a single massless scalar is not standard and may even look impossible at first sight.

Both for BCS superconductivity and for anyon superconductivity (which may or may not be realized in nature in the new high T_c materials), the relation (1.1) is limited to idealized models, not real materials. Nevertheless, understanding the consequences at low energies of a relation such as (1.1) is of methodological interest. Also, the investigation of the consequences of the simplest idealized models of BCS and anyon superconductors is illuminating. This may be self-evident in the case of the novel anyon superconductors; but actually, even in the more familiar BCS case, there is something to be learned by studying the consequences of (1.1), as we will now discuss.

The conventional Hall effect arises if a conductor is placed in a magnetic field in, say, the z direction, with a current flowing in, say, the y direction. Because of the $v \times B$ force on the current carriers, there results an electric polarization of the material in the x direction and an electric potential gradient in that direction. This, at any rate, is the familiar story in an ordinary conductor.

One might believe that in a superconductor, this potential gradient would be shorted out and the Hall effect would be absent. Nevertheless, London[8] argued nearly half a century ago that a Hall-like effect would occur in a superconductor, though with a different physical mechanism. He considered what we would now call a Type-I superconductor, with electric currents carried in a small boundary layer, and argued that when a current J is flowing, there would be a potential drop across the current carrying layer, proportional to J^2. (Since the magnetic field B present in this situation is proportional to J, the potential drop is proportional to $J \times B$ just as in the standard Hall effect in an ordinary conductor.) We will call this effect the London Hall effect.

London's reasoning was based on a hydrodynamic model of superconductors, which influenced the modern understanding in terms of spontaneously broken symmetry but does not coincide with it. One of our goals in this paper is to clarify the realm of validity of London's reasoning, as judged from a modern standpoint. We will argue that the additional input required to justify London's formula for the London Hall effect is precisely (1.1). This relation is a precise mathematical statement of a link between the particle current and the momentum density which justifies London's considerations of quantum hydrodynamics. To ensure the validity of (1.1), the effective Lagrangian must contain certain non-minimal terms with definite coefficients; these terms lead to the London Hall effect. For a realistic BCS superconductor, (1.1) is not valid so that London's formula will not be precisely accurate; but one still expects a London Hall effect of the same order of magnitude, since the relevant non-minimal terms should be present with coefficients of the same order of magnitude.

The contents of the paper are as follows. In Sec. 2 we discuss Lagrangian realizations of (1.1), both as an exact relation and as an approximate relation in the presence of electromagnetism. Some aspects of time-dependent London and Landau-Ginzburg theory are clarified. In Sec. 3 we discuss the London Hall effect in superconductors, and show using (1.1) that its magnitude is determined in terms of the effective mass of the condensed species. London originally thought this effect would be too small to measure. With advances in technology, however, it has become accessible to observation.[9] In Sec. 4 we discuss two applications of the same circle of ideas to the phenomenology of anyon superconductors. First: there is a Hall effect linear in the current (at zero external magnetic field) whose fundamental nature we clarify and emphasize. Second: the hydrodynamic identity (1.1), applied to the ideal anyon superconductor, leads to a demonstration of the vanishing of the b term.[a] It was in this context that we were originally led to the considerations of this paper. Sections 3 and 4 also contain some remarks on how experiments to measure the relevant effects might be carried out, and on their theoretical significance. Finally in Sec. 5 we briefly consider possible generalizations of the basic idea.

2. Corrections to Effective Lagrangians

We first consider the implementation of (1.1) in the simple case of a real scalar field ϕ. This will be sufficient for the description of longitudinal sound modes. We impose the symmetry

$$\phi \rightarrow \phi + \alpha \tag{2.1}$$

that we shall seek to associate with particle number conservation. The standard Lagrangian to describe sound waves with speed of propagation v is of course

$$L_0 = \frac{\rho}{2m}\left\{\frac{1}{v^2}(\partial_0\phi)^2 - (\partial_i\phi)^2\right\}. \tag{2.2}$$

(The overall multiplicative factor, which of course does not affect the equations of motion, has been chosen for later convenience.) This Lagrangian respects (2.1). However, it does not at all respect (1.1); indeed

$$J_i = e\frac{\delta L}{\delta\partial_i\phi} = -\frac{\rho e}{m}\partial_i\phi \tag{2.3}$$

[a] We follow the notation of Ref. 3.

while

$$T_{0i} = -\frac{\delta L}{\delta \partial_0 \phi} \partial_i \phi = -\frac{\rho}{mv^2} \partial_0 \phi \partial_i \phi. \tag{2.4}$$

Looking at the canonical expressions for J and T, we realize that our fundamental relation $J_i = e/m\ T_{0i}$ will be satisfied identically for Lagrangians of the form

$$L = P\left(\partial_0 \phi - \frac{1}{2m}(\partial_i \phi)^2\right), \tag{2.5}$$

where P is an arbitrary polynomial. The corrected L which, in a sense presently to be made precise, reduces to the simple Lagrangian (2.2) in the long-wavelength limit, is constructed with a quadratic P:

$$L_c = \rho\left(\partial_0 \phi - \frac{1}{2m}(\partial_i \phi)^2\right) + \frac{\rho}{2mv^2}\left(\partial_0 \phi - \frac{1}{2m}(\partial_i \phi)^2\right)^2. \tag{2.6}$$

Expanding this expression, we find

$$L_c = \rho \partial_0 \phi + L_0 - \frac{\rho}{2m^2 v^2} \partial_0 \phi (\partial_i \phi)^2 + \frac{\rho}{8m^3 v^2} (\partial_i \phi)^2 (\partial_j \phi)^2. \tag{2.7}$$

The first term on the right-hand side is a total divergence, and does not contribute to the equations of motion. Nevertheless it is not devoid of physical meaning. Indeed, it contributes the constant value ρe to the density J_0. This represents the charge density when the fluid is uniform and static. Its value explains our choice of normalization.

The last two terms on the right-hand side are true dynamical corrections to L_0; they certainly do alter the equations of motion. They may be neglected if ϕ is small or if it is sufficiently smoothly varying in space and time. In general, however, they must be included.

For later use let us record the full corrected expression for the charge density:

$$J_0 = \rho e + \frac{\rho e}{mv^2} \partial_0 \phi - \frac{\rho e}{2m^2 v^2} (\partial_i \phi)^2. \tag{2.8}$$

Now let us consider the case of a complex scalar field ψ, with the symmetry

$$\psi \rightarrow e^{i\alpha} \psi. \tag{2.9}$$

The standard non-relativistic Lagrangian for this field, leading to the dispersion relation $E = p^2/2m$ between energy and momentum, is

$$L_0 = \frac{-i}{2}(\psi^* \partial_0 \psi - \partial_0 \psi^* \psi) - \frac{1}{2m} \partial_i \psi^* \partial_i \psi. \tag{2.10}$$

The charge density and current are

$$J_0 = ie\left\{\frac{\delta L}{\delta \partial_0 \psi}\psi - \frac{\delta L}{\delta \partial_0 \psi^*}\psi^*\right\} = e\psi^*\psi \tag{2.11}$$

$$J_i = ie\left\{\frac{\delta L}{\delta \partial_i \psi}\psi - \frac{\delta L}{\delta \partial_i \psi^*}\psi^*\right\} = \frac{ie}{2m}(\psi^* \partial_i \psi - \partial_i \psi^* \psi). \tag{2.12}$$

It is not difficult to see that the current and the canonical momentum density are indeed related according to (1.1).

Equation (2.10) — after the "minimal coupling" replacement $\partial_\mu \rightarrow \partial_\mu + eA_\mu$ — is sometimes proposed as the effective Lagrangian to describe superconductors.[10] The basis of this proposal is the intuitive argument that one should have a Schrödinger equation for the Cooper pairs; and variation of (2.10) does of course generate precisely the Schrödinger equation for ψ. (In this interpretation, of course, m and e are understood to be *twice* the mass and charge of the electron.) However, this argument cannot be quite correct. The theory of spontaneous symmetry breaking tells us that in the absence of electromagnetism the formation of a condensate of Cooper pairs should lead to the appearance of a Nambu-Goldstone boson field with a linear dispersion relation near zero. The excitations of this field will dominate the low-energy, long-wavelength behavior of the model. They should be described by a Lagrangian like (2.7); i.e. essentially as sound waves.

To obtain a Lagrangian that still satisfies $J_i = e/m\ T_{0i}$ but also behaves properly in the low-energy, long-wavelength limit, we proceed by analogy with our treatment of the real scalar field. It is not difficult to find the relation that replaces (2.5); it is

$$L = P\left(\frac{-i}{2}(\psi^* \partial_0 \psi - \partial_0 \psi^* \psi) - \frac{1}{2m} \partial_i \psi^* \partial_i \psi\right) = P(L_0). \tag{2.13}$$

Again, we expect that a quadratic P should be sufficient. To fix the coefficient of the square term, let us consider how a Nambu-Goldstone mode described by (2.7) will be obtained. It should correspond to the *ansatz*

$$\psi = Ve^{i\phi} \tag{2.14}$$

describing a space-time dependent symmetry operation on the condensate. A simple calculation suffices to show that taking

$$L_c = L_0 + \frac{1}{2mv^2\rho} L_0{}^2 \tag{2.15}$$

does the job, with $V = \sqrt{\rho}$.

In this way, we are able to accommodate the fundamental relation (1.1) in a Lagrangian sufficiently flexible to describe the dynamics of a simple Nambu-Goldstone field, and also to allow for variations in the magnitude of the order parameter. In our opinion, a Lagrangian such as (2.15) is (after the replacement of ordinary by gauge covariant derivatives) the proper starting point for the discussion of time-dependent phenomena in the Landau-Ginzburg framework.[11] It should be remarked, in this connection, that the constraint $J_i = e/m\, T_{0i}$ is empty when applied to non-derivative couplings.

Although our main focus shall be on the implications of the correction terms we have found above for superconductors, we shall now make a few brief remarks concerning their physical interpretation and implications for neutral systems.

i) Consider first Eq. (2.8) for the charge density. The new gradient term implies that even for stationary flow, with all time-derivatives vanishing, the density varies. The density is least where the current is greatest. This phenomenon reflects the physics of Bernoulli's principle: in a rapid-flow region there is low pressure, and thus low density.

ii) The prescription $\partial_0 \to \partial_0 - 1/2m\,(\partial_i\phi)\partial_i$ that results from (2.5) is highly reminiscent of the usual prescription $\partial_t \to \partial_t + (v \cdot \nabla)$ for the convective derivative. (There is a peculiar factor of two discrepancy.) This suggests that our corrections embody the effects of inertia. This suggestion is reinforced by the fact that the corrections vanish in the limit $m \to \infty$. Our corrections become quantitatively significant for the bulk flow when the speed associated with the particles becomes comparable with the speed of sound, or precisely when

$$\frac{1}{2m}\partial_i\phi \sim v. \tag{2.16}$$

Of course, there are other corrections to the simple equations for the propagation of sound in this regime, especially (in an ordinary gas) due to viscosity. We have not carefully investigated whether there are regimes in which our corrections are the dominant ones, although this would seem to be plausible for superfluids.

Significant too is the fact that (2.5) is essentially the *unique* solution to

$$-\frac{\delta L}{\delta\partial_0\phi}\partial_i\phi = m\frac{\delta L}{\delta\partial_i\phi}. \tag{2.17}$$

This means that it is difficult to incorporate gradient terms other than those arising from the modification of ∂_0, i.e. from convection. (We have not carefully considered possible non-canonical modifications of T_{0i} and J_i, so we cannot completely rule out the possibility that (1.1) may be solved in other ways.) The vanishing of the b term in anyon superconductivity follows from this consideration, as we shall elaborate in Sec. 4.
iii) The fact that the correction are non-linear means that they induce scattering and decay of sound waves. Again, we have not attempted to investigate under what circumstances, if any, this is a dominant mechanism.

3. The London Hall Effect

When the conserved current J is coupled to the electromagnetic field, relation (1.1) can no longer be exact. This is because the electromagnetic field carries momentum but does not contribute to the electromagnetic current. Nevertheless we expect that a relation similar to (1.1) will be approximately valid, because of the smallness of the electromagnetic coupling constant, for weak fields. Thus we shall assume that there is a relation of the form

$$J_i = \frac{e}{m} T_{0i}^{\text{mat.}} \tag{3.1}$$

where $T_{0i}^{\text{mat.}}$ is identified as the matter part of the momentum density. We require that $T_{0i}^{\text{mat.}}$ becomes the complete energy-momentum tensor as the electromagnetic field strength approaches zero. We also require, naturally, that it be gauge invariant.

Given the discussion of the previous section, it is not difficult to construct effective Lagrangians embodying (3.1). Indeed, it is more or less obvious that the correct procedure must be simply to replace ordinary by gauge covariant derivatives in the Lagrangians derived there. In fact this is correct, but two subtleties arise that ought to be mentioned:
i) The canonical momentum density (2.4) is not gauge invariant. Neither is the corresponding density for the electromagnetic field:

$$T_{0i}^{\text{em,can.}} = -\frac{\delta L}{\delta \partial_0 A_j} \partial_i A_j. \tag{3.2}$$

The reason for this is simply that the canonical density generates translations. Under a translation of the spatial coordinates, $x^\mu \rightarrow x^\mu + a^\mu$, one has naively

$$\phi \rightarrow \phi + a_\mu \partial_\mu \phi \tag{3.3}$$

$$A_\nu \rightarrow A_\nu + \lambda a_\mu \partial_\mu A_\nu, \tag{3.4}$$

and the differences that appear on the right-hand side are not gauge invariant. To repair this situation, we employ our freedom to supplement the naive translation with a suitable gauge transformation

$$\phi \to \phi - ef \tag{3.5}$$

$$A_\mu \to A_\mu + \partial_\mu f. \tag{3.6}$$

Making the choice

$$f = -a_\mu A_\mu \tag{3.7}$$

we find that (3.5), (3.6) are replaced by the manifestly covariant equations

$$\phi \to \phi + a_\mu D_\mu \phi \tag{3.8}$$

$$A_\nu \to A_\nu + a_\mu F_{\mu\nu}, \tag{3.9}$$

where D is the covariant derivative and F the field strength. The generators of the modified spatial translations are then the gauge invariant forms

$$T_{0i}^{\text{mat.}} = -\frac{\delta L}{\delta \partial_0 \phi} D_i \phi = -\frac{\delta L}{\delta D_0 \phi} D_i \phi, \tag{3.10}$$

$$T_{0i}^{\text{em}} = -\frac{\delta L}{\delta \partial_0 A_\nu} F_{i\nu} = -\frac{\delta L}{\delta F_{0\nu}} F_{i\nu}. \tag{3.11}$$

It is in terms of this form of $T_{0i}^{\text{mat.}}$ that we require (3.1).

Similar procedures, of course, apply to the complex scalar field.

ii) After replacement of ordinary by gauge covariant derivatives, the linear term $\partial_0 \phi \to D_0 \phi$ ceases to be a total derivative. Thus it would appear that this term, which is universally ignored, contributes to the equations of motion. However, a neutralizing uniform background contributes

$$\Delta L = -\rho e A_0 \tag{3.12}$$

to the Lagrangian. The effect of such a background is precisely to undo the passage from ordinary to covariant derivative in the linear term, and thus to restore it to its previous inconsequential status.

Now we turn to a discussion of the London Hall effect, that is, London's prediction that when a current flows in a superconductor, an electric potential develops across it. One interpretation of the effect is that equilibrium requires that the *electrochemical* potential should be constant in a superconductor. When current is flowing the chemical potential is not constant, and an electrostatic potential develops to compensate for it.

Our discussion will be entirely confined to the vortex-free, non-dissipative Meissner regime.

The London Hall potential is quadratic in the current, consistent with time-reversal symmetry, since a time-reversal transformation reverses the current but not the potential. In view of the link between magnetic field and current for a superconductor in the Meissner regime, we may also say that the London potential is jointly proportional to the current and magnetic field (thus making the connection with the ordinary Hall effect), or quadratic in the magnetic field.

We shall first derive the most important equation heuristically, and then justify it with more careful arguments. It is simplest to begin with the expression (2.8) for the charge density, modified to include the coupling to electromagnetism and with the constant piece cancelled off by the uniform background:

$$J_0 = \rho e\left(\frac{1}{mv^2} D_0\phi - \frac{1}{2m^2v^2}(D_i\phi)^2\right). \tag{3.13}$$

It is plausible, and will be argued quantitatively below, that in ordinary circumstances, for quasistatic phenomena, $J_0 = 0$ to a good approximation. This simply reflects that Coulomb repulsion dominates the energetics. Assuming it for the moment, we have

$$D_0\phi = \frac{1}{2m}(D_i\phi)^2. \tag{3.14}$$

Generally in considering the electrodynamics of (simply connected) superconductors one works in "London gauge" $\phi = 0$; that this really is a gauge choice follows from (3.5). Making this choice, and further using the approximate expression $J_i \sim -\rho e/m\, D_i\phi$ for the current we find immediately from (3.14) a relation between electric potential and current:

$$A_0 = \frac{m}{2\rho^2 e^3} J_i^2. \tag{3.15}$$

This is the essence of the London Hall effect.

Equation (3.15) appears rather peculiar, because it refers explicitly to the gauge-dependent quantity A_0. Of course this is not a contradiction, because (3.14) was derived in a particular gauge; but it makes it incumbent on us to discuss its precise physical meaning. The sort of experiment one might do to test (3.15) is to put capacitor plates close to but not in contact with two locations in the superconducting sample where different amounts of supercurrent flow, and then to measure the potential difference across these plates by monitoring the charge flow induced in the exterior circuit as the supercurrent flow varies. To the extent that the superconductor may be treated quasistatically, we should be able to find a solution for the current flow in it such that all the fields are static, and in particular $\partial_0\phi = 0$ and $\partial_0 A_i = 0$. The first of these equalities justifies the passage from (3.14) to (3.15); the second allows us to interpret the difference between A_0 measured at two points within the superconductor as the integral of the electric field along any path connecting them. Thus (3.15), despite its reference to gauge-dependent quantities, has a clear (gauge invariant) observable meaning.

Now we shall analyze a specific realization of the London Hall effect. Consider a semi-infinite slab of superconductor occupying the half-space $x > 0$ and subject to an external magnetic field in the z-direction of magnitude B at $x = 0$. We look for a stationary solution in London gauge, taking as our *ansatz* that only A_0 and A_y are non-zero, and that these depend only on x. The Maxwell-London equations may be derived by varying our effective Lagrangian, supplemented of course with the standard Maxwell Lagrangian. The equations read:

$$-\partial_x^2 A_0 = 4\pi\left(\frac{\rho e^2}{mv^2}A_0 - \frac{\rho e^3}{2m^2v^2}A_y^2\right), \tag{3.16}$$

$$-\partial_x^2 A_y = \frac{4\pi}{c^2}\left(-\frac{\rho e^2}{m}A_y - \frac{\rho e^3}{m^2v^2}A_y\left(A_0 - \frac{e}{2m}A_y^2\right)\right). \tag{3.17}$$

To solve these we make the trial approximation suggested above:

$$A_0 - \frac{e}{2m}A_y^2 = 0 \tag{3.18}$$

and solve (3.17); the result is

$$A_y = \frac{\lambda B}{c}e^{-x/\lambda} \tag{3.19}$$

where

$$\lambda^2 = \frac{mc^2}{4\pi\rho e^2} \tag{3.20}$$

is the square of the standard London penetration depth. (Notice that with our conventions B is $-c$ times the curl of A.) The electric potential is given by

$$A_0 = \frac{e}{2m}\frac{\lambda^2 B^2}{c^2} e^{-2x/\lambda}. \tag{3.21}$$

Notice that it penetrates only half the distance of the magnetic field.

Now we can check the accuracy of our assumed approximation (3.18). Equation (3.16) is not precisely satisfied; however the ratio of the residual term to the terms kept is

$$\frac{\partial_x^2 A_0}{4\pi \frac{\rho e^2}{m v^2} A_0} = 4\frac{v^2}{c^2} \ll 1. \tag{3.22}$$

The characteristic parameter of smallness for the terms neglected is essentially the square of the ratio of the electric to magnetic penetration depths, or equivalently the square of the ratio of the speed of sound to the speed of light.

For the electrostatic potential difference between the edge of the sample and the interior we find

$$\Delta A_0 = \frac{1}{4\pi \rho e} B^2. \tag{3.23}$$

Thus the London Hall effect gives a direct measure of the superfluid charge density.

In this connection, let us note by way of contrast that the penetration depth is sensitive to the ratio of superfluid charge density to carrier mass — indeed, only this ratio appears as a parameter in the effective Lagrangian, if the higher-gradient terms we have been emphasizing in this paper are neglected. Thus in the truncated effective theory neither the density nor the mass have individual meaning. In his excellent book Tinkham[12] quotes de Gennes as saying the effective mass could be taken to be the mass of the sun! Consideration of the London Hall effect together with the penetration depth, furnishes us with definite independent observable meanings for the mass and the density.

Numerically, if we put $\rho = 10^{21}/\text{cm}^3$, $B = 50$ gauss then we find $\Delta A_0 = 1.2 \times 10^{-7}$ volts. These are representative values for the high-temperature superconductors. For conventional superconductors both ρ and B can be larger. Because of the situation mentioned in the previous paragraph, there would seem to be considerable interest in the experimental measurement of the London Hall effect, especially in strong-coupling or exotic superconductors where the effective mass may differ considerably from the bare electron mass.

4. Applications to Anyon Superconductivity

In this section we shall consider two issues that arise in the phenomenology of anyon superconductivity.

The most characteristic feature of the effective Lagrangian derived in Ref. 3 for the description of the electrodynamics of anyon superconductors is the possible appearance of terms that violate the discrete symmetries under spatial parity P and time reversal T (while leaving PT as a good symmetry).

If for the moment we do not worry about enforcing (1.1), these terms may be included into the effective Lagrangian by adding

$$\Delta L = e\alpha D_0\phi\varepsilon_{ij}F_{ij} + e\beta D_i\phi\varepsilon_{ij}F_{0j} \tag{4.1}$$

to it. Here the indices i, j run over the two directions tangent to the plane in which the anyons move. The Lagrangian in (4.1), and the currents and densities below, are to be interpreted as two-dimensional — they describe the dynamics in a single plane. Our α, β here are proportional to the a and b used in Ref. 3.

The additional terms modify the charge density by

$$\Delta J_0 = e^2\alpha\varepsilon_{ij}F_{ij}. \tag{4.2}$$

Thus Eqs. (3.14), (3.15) are modified to read

$$D_0\phi = \frac{1}{2m}(D_i\phi)^2 - e\alpha\frac{mv^2}{\rho}\varepsilon_{ij}F_{ij}, \tag{4.3}$$

$$A_0 = \frac{m}{2\rho^2 e^3}J_i^2 - \frac{mv^2}{\rho}\alpha\varepsilon_{ij}F_{ij}. \tag{4.4}$$

We see that in addition to the London Hall effect there is an additional contribution to the potential of a new sort, let us call it the linear Hall effect, that is *linear* in the field or current and therefore manifestly violates P and T. The magnitude of the coefficient α can be computed from the microscopic anyon model of Refs. 3, 13; in the notation used there we find

$$\alpha = -\frac{aC}{2e^2} = \left(n - \frac{1}{n}\right)\frac{1}{16\pi} \tag{4.5}$$

for anyons with statistics $\theta = \pi(1 - 1/n)$.

In Ref. 3 the linear Hall effect was computed in the large n limit for a special geometry, and the result was expressed in terms of the ratio of potential difference to total current. In our opinion, Eq. (4.4) brings out the essential physics much more clearly — the linear Hall effect is a simple proportionality between electric potential and magnetic field. Furthermore, the coefficient α, or $aC/2e^2$ in the previous notation, is a quantity that can be related to very fundamental properties of the angular momentum in anyon models (it is basically the intrinsic orbital moment). One sign of this is that it has dimensions of an action; indeed in (4.5) we find that no continuous material parameters occur. It has been calculated, using arguments that appear to be exact, not only in the one-species anyon model in the large n limit but also in the multi-species models more likely to arise from underlying lattice systems.[13,14]

Numerically, if we put $B = 50$ gauss as before, and take m equal to the electron mass, we find $\Delta A_0 = 2.9 \times 10^{-7}$ volts. Thus for these parameter values the linear Hall effect is slightly larger than the London Hall effect.

Now let us consider the consequences of enforcing (1.1). For the term proportional to α, the effect is minimal: we must replace $D_0\phi$ by $D_0\phi - 1/2m \times (D_i\phi)^2$. This change has a small quantitative effect, but does not change the overall picture significantly. The term proportional to β, on the other hand, is simply forbidden — one cannot allow terms linear in $D_i\phi$. This accords with the conclusion reached by the use of (essentially equivalent) physical arguments in Refs. 3, 13.

5. Possible Extensions

There are several directions in which this work might be extended. Perhaps the most obvious is to consider cases where there is more than one particle species contributing to the momentum flow. An example that has received considerable experimental attention is dilute solutions of ^{3}He ions in ^{4}He. The appropriate generalization of (1.1) is simply

$$T_{0i} = m_1 J_i^{(1)} + m_2 J_i^{(2)} \tag{5.1}$$

where $J^{(1)}$ and $J^{(2)}$ are the particle number currents for the two species and m_1, m_2 are the corresponding masses. Building on Sec. 2, we introduce fields ϕ and ψ to represent the superfluid and the dressed ion quasiparticles respectively, and assume that the currents appearing in (5.1) are associated with the symmetries

$$\phi \to \phi + \alpha \tag{5.2}$$

$$\psi \to e^{i(\eta\alpha + \beta)}\psi \tag{5.3}$$

η is a parameter that, roughly speaking, tells us how many ^{4}He atoms are carried in the ion quasiparticle cloud. Then (5.1) will constrain the effective Lagrangian for ϕ and ψ as follows:

$$-\frac{\delta L}{\delta\partial_0\phi}\partial_i\phi-\frac{\delta L}{\delta\partial_0\psi}\partial_i\psi-\frac{\delta L}{\delta\partial_0\psi^*}\partial_i\psi^*=m_1\frac{\delta L}{\delta\partial_i\phi}+i(\eta m_1+m_2)\left(\frac{\delta L}{\delta\partial_i\psi}\psi-\frac{\delta L}{\delta\partial_i\psi^*}\psi^*\right). \tag{5.4}$$

Now (5.4) is satisfied by the expression

$$\partial_0\phi-\frac{1}{2m}(\partial_i\phi)^2 \tag{5.5}$$

and is also satisfied by the expression

$$-i(\eta m_1+m_2)(\psi^*\partial_0\psi-\partial_0\psi^*\psi)-b(\partial_i\psi^*)(\partial_i\psi). \tag{5.6}$$

An appropriate effective Lagrangian can be constructed using linear and quadratic functions of these objects. The resulting Lagrangian will — as in our discussion of one fluid models — inevitably contain non-minimal terms, which appear with universal coefficients.

Thus we find that we do obtain constraints on the effective Lagrangian. If the various coefficients could be measured experimentally, for liquid helium mixtures, they would among other things answer the rather intriguing question: how many ^{4}He atoms are entrained in the ^{3}He quasiparticle cloud?

One might also consider ^{4}He itself, at finite temperatures, so that both superfluid and rotons are present. Whereas in the previous case we had two separate conserved particle number currents, in the present case there is just one, which is (to first approximation) a linear combination of superfluid and roton pieces.

A potentially rich but less straightforward extension would be to effective Lagrangians containing more complicated fields, such as are used to describe the superfluid phases of ^{3}He.

Returning to anyon superconductivity, it is interesting that the coefficient α, which as we saw seems to be rather fundamental, can be characterized as the coefficient of proportionality between the *low energy and momentum limit* of the charge density and magnetic field:

$$J_0\rightarrow\alpha e^2\varepsilon_{ij}F_{ij}. \tag{5.7}$$

This relation, like (1.1), should carry over from the underlying microscopic theory into the effective macroscopic theory.

Our considerations thus far have all concerned non-relativistic systems. Is it possible to constrain relativistic systems by operator identities among symmetry generators? Simple considerations of index-matching convince us that the possibilities for *linear* relations are quite limited in standard four-dimensional theories. One can imagine imposing relations like (1.1) in the context of Kaluza-Klein theories, with the 0 in T_{0i} replaced by some compactified direction (and i running from 0 to 3). Finally, let us observe that the famous Sugawara *ansatz*

$$T_{\mu\nu} = j_\mu j_\nu \tag{5.8}$$

is a relativistic constraint in the same family, but of course non-linear.

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